Geometry of the reduced quantum plane[*]

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Abstract

We consider the space $\mathcal{M}$ of $N \times N$ matrices as a reduced quantum plane and discuss its geometry under the action and coaction of finite dimensional quantum groups (a quotient of $U_q SL(2)$, q being an N-th root of unity, and its dual). We also introduce a differential calculus for $\mathcal{M}$: a quotient of the Wess Zumino complex. We shall restrict ourselves to the case N odd and often choose the particular value $N = 3$.

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1 Introduction

When q is a root of unity ($q^N = 1$), the quantized enveloping algebra $U_q sl(2, \mathbb{C})$ possesses interesting quotients that are finite dimensional Hopf algebras. The structure of the left regular representation of such an algebra was investigated in [1] and the pairing with its dual in [2]. We call $\mathcal{H}$ the Hopf algebra quotient of $U_q sl(2, \mathbb{C})$ defined by the relations $K^N = \mathbb{1}$, $X_\pm^N = 0$ (we shall define the generators $K, X_\pm$ in a later section), and $\mathcal{F}$ its dual. It was shown[1] in [1] that the non semi-simple algebra $\mathcal{H}$ is isomorphic with the direct sum of a complex matrix algebra and of several copies of suitably defined matrix algebras with coefficients in the ring $Gr(2)$ of Grassmann numbers with two generators. The explicit structure (for all values of N) of those algebras, including the expression of generators themselves, in terms of matrices with coefficients in $\mathbb{C}$ or $Gr(2)$ was obtained by [3]. Using these results, the representation theory of $\mathcal{H}$, for the case $N = 3$, was presented in [4]. Following this work, the authors of [5] studied the action of $\mathcal{H}$ (case $N = 3$) on the algebra of complex matrices $\mathcal{M} \equiv M_3(\mathbb{C})$. In the letter [6] a reduced Wess-Zumino complex $\Omega_{WZ}(\mathcal{M})$ was introduced, thus providing a differential calculus bicovariant with respect to the action of the quantum group $\mathcal{H}$ on the algebra $M_3(\mathbb{C})$ of complex matrices. This differential algebra (that could be used to generalize gauge field theory models on an auxiliary smooth manifold) was also analysed in terms of representation theory of $\mathcal{H}$ in the same work. In particular, it was shown that $M_3(\mathbb{C})$ itself can be reduced into the direct sum of three indecomposable representations of $\mathcal{H}$. This result was generalized in [7]. A general discussion of several other properties of the dually paired Hopf algebras $\mathcal{F}$ and $\mathcal{H}$ (scalar products, star structures, twisted derivations, etc.) can also be found there, as well as in the article [8]. In the present contribution we present a summary of results already discussed in the papers [6, 8, 7]. The original purpose of our work was to define generalized differential forms and generalized connections on a smooth manifold, with values in the Lie algebra of the linear group $GL(N)$, in such a way that there would be a non trivial global action of some Hopf algebra on the so obtained differential complex, extending somehow the usual action of the isometry group of the manifold (when it exists) by some internal quantum symmetry. This construction will be recalled in the text.

2 The space $\mathcal{M}$ of $N \times N$ complex matrices as a reduced quantum plane

The algebra of $N \times N$ matrices can be generated by two elements x and y with relations:

$$xy = qyx \qquad \text{and} \qquad x^N = y^N = \mathbb{1} \, , \tag{1}$$

where q denotes an N-th root of unity ($q \neq 1$) and $\mathbb{1}$ denotes the unit matrix. Explicitly, x and y can be taken as the following matrices:

$$x = \begin{pmatrix} 1 & & & & \\ & q^{-1} & & & \\ & & q^{-2} & & \\ & & & \ddots & \\ & & & & q^{-(N-1)} \end{pmatrix} \qquad y = \begin{pmatrix} 0 & & & & \\ \vdots & & & \mathbb{1}_{N-1} & \\ 0 & & & & \\ 1 & 0 & \cdots & & 0 \end{pmatrix} \tag{2}$$

This result can be found in [9].
Warning: for technical reasons, we shall assume in all this paper that N is odd and that q is a *primitive* root of unity.

Here and below, we shall simply denote the algebra $M_N(\mathbb{C})$ by the symbol $\mathcal{M}$, and call it "the reduced quantum plane".

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[1] Warning: the authors of [1] actually consider a Hopf algebra quotient defined by $K^{2N} = 1$, $X_\pm^N = 0$, so that their algebra is, in a sense, twice bigger than ours.

3 The finite dimensional quantum group $\mathcal{F}$ and its coaction on $\mathcal{M}$

One introduces *non-commuting symbols* a, b, c and d generating an algebra $\mathcal{F}$ and imposes that the quantities x', y' (and $\tilde{x}, \tilde{y}$) obtained by the following matrix equalities should satisfy the same relations as x and y.

$$\Delta_L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} \doteq \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad \text{left coaction} \ , \tag{3}$$

and

$$\Delta_R \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \doteq \begin{pmatrix} \tilde{x} & \tilde{y} \end{pmatrix} \qquad \text{right coaction} \ . \tag{4}$$

These equations also define left and right coactions, that extend to the whole of $\mathcal{M}$ using the homomorphism property $\Delta_{L,R}(fg) = \Delta_{L,R}(f)\Delta_{L,R}(g) \quad (f, g \in \mathcal{M})$. Here one should not confuse Δ (the coproduct on a quantum group that we shall introduce later) with $\Delta_{R,L}$ (the R, L-coaction on $\mathcal{M}$)!

The elements a, b, c, d should therefore satisfy an algebra such that

$$\Delta_L(xy - qyx) \quad = \quad 0 \tag{5}$$
$$\Delta_L(x^N - 1) = \Delta_L(y^N - 1) \quad = \quad 0 \ , \tag{6}$$

and the same for Δ_R. This leads to the usual relations defining the algebra of "functions over the quantum plane" [10]:

$$\begin{array}{ll} qba = ab & qdb = bd \\ qca = ac & qdc = cd \\ cb = bc & ad - da = (q - q^{-1})bc \ , \end{array} \tag{7}$$

but, we also have non quadratic relations:

$$\begin{array}{ll} a^N = \mathbb{1} \ , & b^N = 0 \ , \\ c^N = 0 \ , & d^N = \mathbb{1} \ . \end{array} \tag{8}$$

The element $\mathcal{D} \doteq da - q^{-1}bc = ad - qbc$ is central (it commutes with all the elements of $Fun(GL_q(2))$); it is called the q-determinant and we set it equal to $\mathbb{1}$. Since $a^N = \mathbb{1}$, multiplying the relation $ad = \mathbb{1} + qbc$ from the left by $a^{(N-1)}$ leads to

$$d = a^{(N-1)}(\mathbb{1} + qbc) \ , \tag{9}$$

so d is not needed and can be eliminated. The algebra $\mathcal{F}$ can therefore be *linearly* generated —as a vector space— by the elements $a^\alpha b^\beta c^\gamma$ where indices α, β, γ run in the set $\{0, 1, \ldots, N-1\}$. We see that $\mathcal{F}$ is a *finite dimensional* associative algebra, whose dimension is

$$\dim(\mathcal{F}) = N^3 \ .$$

$\mathcal{F}$ is not only an associative algebra but a Hopf algebra, with the corresponding maps defined on the generators as follows:

Coproduct: $\Delta a = a \otimes a + b \otimes c$, $\Delta b = a \otimes b + b \otimes d$, $\Delta c = c \otimes a + d \otimes c$, $\Delta d = c \otimes b + d \otimes d$.

Antipode: $Sa = d$, $Sb = -q^{-1}b$, $Sc = -qc$, $Sd = a$.

Counit: $\epsilon(a) = 1$, $\epsilon(b) = 0$, $\epsilon(c) = 0$, $\epsilon(d) = 1$.

We call $\mathcal{F}$ the *reduced quantum unimodular group* associated with an N-th root of unity. It is, by construction, an associative algebra. However, it is not semi-simple. Therefore, $\mathcal{F}$ is not a matrix quantum group in the sense of Woronowicz [11].

The coaction of $\mathcal{F}$ on $\mathcal{M}$ was given above. Actually, $\mathcal{M}$ endowed with the two coactions Δ_L and Δ_R is a left and right comodule *algebra* over $\mathcal{F}$, i.e., a corepresentation space of the quantum group $\mathcal{F}$ such that

$$\begin{array}{rcl} \Delta_{L,R}(zw) & = & \Delta_{L,R}(z) \, \Delta_{L,R}(w) \\ \Delta_{L,R}(\mathbb{1}) & = & \mathbb{1} \otimes \mathbb{1} \ . \end{array} \tag{10}$$

4 The dual $\mathcal{H}$ of $\mathcal{F}$, and its action on $\mathcal{M}$

Being $\mathcal{F}$ a quantum group (a Hopf algebra), its dual $\mathcal{H} \doteq \mathcal{F}^*$ is a quantum group as well. Let $u_i \in \mathcal{F}$ and $X_i \in \mathcal{H}$. We call $< X_i, u_j >$ the evaluation of X_i on u_j (a complex number).

- Starting with the coproduct Δ on $\mathcal{F}$, one defines a product on $\mathcal{H}$, by $< X_1 X_2, u > \doteq < X_1 \otimes X_2, \Delta u >$.

- Using the product in $\mathcal{F}$, one defines a coproduct (that we again denote Δ) in $\mathcal{H}$ by $< \Delta X, u_1 \otimes u_2 > \doteq < X, u_1 u_2 >$.

- The interplay between unit and counit is given by: $< \mathbb{1}_{\mathcal{H}}, u > = \epsilon_{\mathcal{F}}(u)$ and $< X, \mathbb{1}_{\mathcal{F}} > = \epsilon_{\mathcal{H}}(X)$.

The two structures of algebra and coalgebra are clearly interchanged by duality.

It is clear that $\mathcal{H}$ is a vector space of dimension N^3. It can be generated, as a complex algebra, by elements $X_\pm$, K dual to the generators of $\mathcal{F}$:

$$< K, a >= q \qquad < X_+, b >= 1 \qquad < X_-, c >= 1 \ ,$$

all other pairings between generators being zero. In this way we get:

Multiplication:

$$
\begin{aligned}
K X_\pm &= q^{\pm 2} X_\pm K \\
[X_+, X_-] &= \frac{1}{(q - q^{-1})}(K - K^{-1}) \\
K^N &= \mathbb{1} \\
X_+^N = X_-^N &= 0 \ .
\end{aligned}
\tag{11}
$$

Comultiplication:

$$
\begin{aligned}
\Delta X_+ &= X_+ \otimes \mathbb{1} + K \otimes X_+ \\
\Delta X_- &= X_- \otimes K^{-1} + \mathbb{1} \otimes X_- \\
\Delta K &= K \otimes K \\
\Delta K^{-1} &= K^{-1} \otimes K^{-1} \ .
\end{aligned}
\tag{12}
$$

It extends to the whole $\mathcal{H}$ as an algebra morphism, i.e., $\Delta(XY) = \Delta X \, \Delta Y$.

Antipode: It is defined by: $S\mathbb{1} = \mathbb{1}$, $SK = K^{-1}$, $SK^{-1} = K$, $SX_+ = -K^{-1}X_+$, $SX_- = -X_-K$, and it extends as an anti-automorphism, i.e., $S(XY) = SY \, SX$. As usual, the square of the antipode is an automorphism, given by $S^2 u = K^{-1} u K$.

Counit: The counit ϵ is defined by $\epsilon\mathbb{1} = \epsilon K = \epsilon K^{-1} = 1$, $\epsilon X_+ = \epsilon X_- = 0$.

Warning: When $q^N = 1$, one can also factorize the universal algebra over the relations $K^{2N} = \mathbb{1}$, $X_\pm^N = 0$, rather than $K^N = \mathbb{1}$, $X_\pm^N = 0$. These relations also define a Hopf ideal but the obtained Hopf algebra is twice as big as ours (K^N is then a central element but is not equal to $\mathbb{1}$).

4.1 Action of $\mathcal{H}$ on $\mathcal{M}$

Using the fact that $\mathcal{F}$ coacts on $\mathcal{M}$ in two possible ways, and that elements of $\mathcal{H}$ can be interpreted as distributions on $\mathcal{F}$, we obtain two commuting actions of $\mathcal{H}$ on the quantum space $\mathcal{M}$. We shall describe the left action for arbitrary elements and give explicit results for the generators.

Let $z \in \mathcal{M}$, $X \in \mathcal{H}$, and $\Delta_R z = z_\alpha \otimes v_\alpha$ with $z_\alpha \in \mathcal{M}$ and $v_\alpha \in \mathcal{F}$ (implied summation). The operation

$$X^L[z] \doteq (id \otimes < X, \cdot >)\Delta_R z = < X, v_\alpha > z_\alpha \ . \tag{13}$$

is a *left* action of $\mathcal{H}$ on $\mathcal{M}$ (dual to the *right*-coaction of $\mathcal{F}$). With this L-action we can check that $\mathcal{M}$ is indeed a left-$\mathcal{H}$-module algebra.

For the case $N = 3$, complete tables are given in [6, 8], and —with other conventions— in [5]. The results can be summarized as follows:
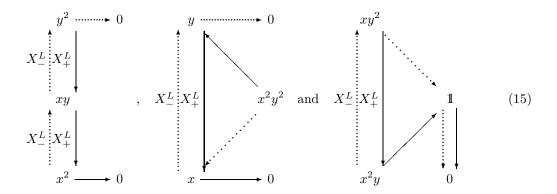
$$
\begin{aligned}
K^L[x^r y^s] &= q^{(r-s)} x^r y^s \\
X_+^L[x^r y^s] &= q^r \left(\frac{1 - q^{-2s}}{1 - q^{-2}}\right) x^{r+1} y^{s-1} \\
X_-^L[x^r y^s] &= q^s \left(\frac{1 - q^{-2r}}{1 - q^{-2}}\right) x^{r-1} y^{s+1}
\end{aligned}
\tag{14}
$$

with $1 \le r, s \le N$.

As $\mathcal{M}$ is a module (a representation space) for the quantum group $\mathcal{H}$, we can reduce it into indecomposable modules. To this end it is necessary to know at least part of the representation theory of $\mathcal{H}$, but note that for the $N = 3$ case we have (up to multiplicative factors)

We see clearly on these diagrams that the algebra of 3×3 matrices can be written as a sum of three inequivalent, 3-dimensional, indecomposable representations of $\mathcal{H}$: an irreducible one, and two indecomposable but reducible modules (each of these two contains a non trivial invariant subspace).

4.2 The structure of the non semisimple algebra $\mathcal{H}$

Using a result by [1], the explicit structure (for all values of N) of those algebras, including the expression of generators $X_\pm, K$ themselves, in terms of matrices with coefficients both in $\mathbb{C}$ and in the Grassmann[2] algebra $Gr(2)$ with two generators θ_1, θ_2, was obtained by [3] (it was explicitly described for $N = 3$ by [4], see also [7] for a general N).

We shall not need the general theory but only the following fact: when N is odd, $\mathcal{H}$ is isomorphic with the direct sum

$$
\mathcal{H} = M_N \oplus \left(M_{N-1|1}(\Lambda^2)\right)_o \oplus \left(M_{N-2|2}(\Lambda^2)\right)_o \oplus \cdots\cdots \oplus \left(M_{\frac{N+1}{2}|\frac{N-1}{2}}(\Lambda^2)\right)_o
\tag{16}
$$

where:

- M_N is an $N \times N$ complex matrix

- An element of the $\left(M_{N-p|p}(\Lambda^2)\right)_o$ part (space that we shall just call $M_{N-p|p}$) is an $(N - p, p)$ block matrix of the following form:

$$
\tag{17}
$$

_____

[2] Remember that $\theta_1^2 = \theta_2^2 = 0$ and that $\theta_1 \theta_2 = -\theta_2 \theta_1$.

We have introduced the following notation:

• is an even element of the ring $Gr(2)$ of Grassmann numbers with two generators, i.e., of the kind $\bullet = \alpha + \beta\theta_1\theta_2$, $\alpha, \beta \in \mathbb{C}$.

○ is an odd element of the ring $Gr(2)$ of Grassmann numbers with two generators, i.e., $\circ = \gamma\theta_1 + \delta\theta_2$, $\gamma, \delta \in \mathbb{C}$.

When N is even, the discussion depends upon the parity of $N/2$ and we shall not discuss this here.

Notice that $\mathcal{H}$ is <u>not</u> a semi-simple algebra: its Jacobson radical $\mathcal{J}$ is obtained by selecting in equation (16) the matrices with elements proportional to Grassmann variables. The quotient $\mathcal{H}/\mathcal{J}$ is then semi-simple... but no longer Hopf!

Projective indecomposable modules (PIM's, also called principal modules) for $\mathcal{H}$ are directly given by the columns of the previous matrices.

- From the M_N block, one obtains N equivalent irreducible representations of dimension N that we shall denote N_{irr}.

- From the $M_{N-p|p}$ block (<u>assume</u> $p < N - p$), one obtains

  – $(N-p)$ equivalent indecomposable projective modules of dimension $2N$ that we shall denote P_{N-p} with elements of the kind

$$(\underbrace{\bullet\,\bullet\cdots\bullet}_{N-p}\underbrace{\circ\,\circ\cdots\circ}_{p})$$

  – p equivalent indecomposable projective modules (also of dimension $2N$) that we shall denote P_p with elements of the kind

$$(\underbrace{\circ\,\circ\cdots\circ}_{N-p}\underbrace{\bullet\,\bullet\cdots\bullet}_{p})$$

Other submodules can be found by restricting the range of parameters appearing in the columns defining the PIM's and imposing stability under multiplication by elements of $\mathcal{H}$. In this way one can determine, for each PIM, the lattice of its submodules. For a given PIM of dimension $2N$ (with the exception of N_{irr}), one finds totally ordered sublattices (displayed below) with exactly three non trivial terms: the radical (here, it is the biggest non trivial submodule of a given PIM), the socle (here it is the smallest non trivial submodule), and one "intermediate" submodule of dimension exactly equal to N. However the definition of this last submodule (up to equivalence) depends on the choice of an arbitrary complex parameter λ, so that we have a chain of inclusions for every such parameter.

4.3 Decomposition of $\mathcal{M} = M_N(\mathbb{C})$ in representations of $\mathcal{H}$

Since there is an action of $\mathcal{H}$ on $\mathcal{M}$, it is clear that $\mathcal{M}$, as a vector space, can be analysed in terms of representations of $\mathcal{H}$. The following result was shown in [7]:

Under the left action of $\mathcal{H}$, the algebra of $N \times N$ matrices can be decomposed into a direct sum of invariant subspaces of dimension N, according to

• $N_N = N_{irr}$: irreducible

• N_{N-1}: reducible indecomposable, with an invariant subspace of dimension $N - 1$.

• N_{N-2}: reducible indecomposable, with an invariant subspace of dimension $N - 2$.

• $\vdots$

• N_1: reducible indecomposable, with an invariant subspace of dimension 1.

The elements of the module called N_p (of dimension N) are of the kind:

$$\underline{N}_p = (\,\gamma_1\theta_{\lambda_1} \quad \gamma_2\theta_{\lambda_2} \quad \cdots \quad \gamma_{N-p}\theta_{\lambda_{N-p}} \quad \beta_1\theta_1\theta_2 \quad \beta_2\theta_1\theta_2 \quad \cdots \quad \beta_p\theta_1\theta_2\,)$$

This submodule is the direct sum of an invariant sub-module of dimension p, and a vector subspace of dimension $N - p$

$$\underline{N}_p = \underline{p} \ \uplus \ (N-p)$$

with

$$\underline{p} = (\,0 \quad 0 \quad \cdots \quad 0 \quad \beta_1\theta_1\theta_2 \quad \beta_2\theta_1\theta_2 \quad \cdots \quad \beta_p\theta_1\theta_2\,)$$

Using these notations, the algebra of complex matrices $N \times N$ can be written

$$\mathcal{M} = N_N \oplus N_1 \oplus N_2 \oplus \cdots \oplus N_{N-1}$$

In the particular case $N = 3$, 3_{odd} is an abelian subalgebra of $\mathcal{M}$, actually isomorphic with the algebra $\mathbb{C}[\mathbb{Z}_3]$ of the abelian group $\mathbb{Z}_3$. Hence, we may write

$$\mathcal{M} = \mathbb{C}[\mathbb{Z}_3] \oplus x\,\mathbb{C}[\mathbb{Z}_3] \oplus x^2\,\mathbb{C}[\mathbb{Z}_3] = 3_{odd} \oplus 3_{eve} \oplus 3_{irr} \ .$$

Moreover, it can be shown that

$$Inverse(3_{eve}) \subset 3_{irr} \ , \quad Inverse(3_{irr}) \subset 3_{eve} \ , \quad \text{but} \quad Inverse(3_{odd}) \subset 3_{odd} \ .$$

4.4   The universal R-matrix

The finite dimensional Hopf algebra $\mathcal{H}$ we have been studying is actually braided and quasi-triangular (as it is well known, the quantum enveloping algebra of $SL(2)$ does *not* possess these properties when q is specialized to a root of unity). The R-matrix of $\mathcal{H}$ can be obtained directly from a general formula given by [13] but we can also get it in a pedestrian way by starting from a reasonable ansatz and imposing certain conditions. Here we take $N = 3$. We start from

$$R = R_K R_X$$

where

$$R_K = \sum_{i,j=0,1,2} c_{ij} K^i \otimes K^j$$

$$R_X = \left[\mathbf{1} \otimes \mathbf{1} + \alpha X_- \otimes X_+ + \beta X_-^2 \otimes X_+^2 \right]$$

Here α, β and the c_{ij} are complex numbers (symmetric in i, j).

A quasi-triangular R matrix should satisfy $(\epsilon \otimes id)R = \mathbf{1}$. As $\epsilon(K) = 1$, this condition implies that

$$\begin{aligned} R_K &= (1 - c_{01} - c_{02})\mathbf{1} \otimes \mathbf{1} + c_{01}(\mathbf{1} \otimes K + K \otimes \mathbf{1}) + c_{02}(\mathbf{1} \otimes K^2 + K^2 \otimes \mathbf{1}) \\ &\quad + c_{12}(K \otimes K^2 + K^2 \otimes K) - (c_{01} + c_{12})K \otimes K - (c_{02} + c_{12})K^2 \otimes K^2 \ . \end{aligned}$$

Also, we should have $(S \otimes S)R = R$. Comparing terms with zero $X_\pm$ grading (i.e., with no $X_\pm$ terms) we find $(S \otimes S)R_K = R_K$, and thus $c_{01} = c_{02}$. Making use of the terms in $X_- \otimes X_+$ we get $c_{01} = 1/3$, $c_{12} = q^2/3$. In the same way, one finds $\alpha = (q - q^2)$ and $\beta = 3q$.

The universal R-matrix now reads explicitly

$$\begin{aligned} R &= \frac{1}{3} \left[\mathbf{1} \otimes \mathbf{1} + (\mathbf{1} \otimes K + K \otimes \mathbf{1}) + (\mathbf{1} \otimes K^2 + K^2 \otimes \mathbf{1}) \right. \\ &\quad \left. + q^2(K \otimes K^2 + K^2 \otimes K) + qK \otimes K + qK^2 \otimes K^2 \right] \\ &\quad \times \left[\mathbf{1} \otimes \mathbf{1} + (q - q^{-1})X_- \otimes X_+ + 3qX_-^2 \otimes X_+^2 \right] \end{aligned} \qquad (18)$$

Using the explicit numerical matrices $X_\pm, K$ given in Appendix E of [8], one can obtain the numerical R matrices in various representations of interest (irreducible or indecomposable ones).

Note that $R^{-1} = R_X^{-1} R_K^{-1}$, where

$$R_X^{-1} = \left[\mathbf{1} \otimes \mathbf{1} - \alpha X_- \otimes X_+ + (\alpha^2 - \beta)X_-^2 \otimes X_+^2 \right]$$

and R_K^{-1} is given by the same expression as R_K but with q and q^2 interchanged. Here we already see that our algebra is *not* triangular, R_{21} (the flipped R) has terms of the form $X_+ \otimes X_-$, whereas R^{-1} only contains terms of the form $X_- \otimes X_+$. It can be straightforwardly verified (but it is cumbersome) that the requirements of almost-cocommutativity and quasi-triangularity hold, namely

$$\Delta^{op}(h) = R\Delta(h)R^{-1} \qquad h \in \mathcal{H} \ , \qquad (19)$$

and

$$
\begin{aligned}
(\Delta \otimes id)(R) &= R_{13}R_{23} \\
(id \otimes \Delta)(R) &= R_{13}R_{12}
\end{aligned}
\tag{20}
$$

For a generic (odd) N the universal R-matrix reads

$$
R = \frac{1}{N}\left[\sum_{0\leq m,n<N} q^{mn}\, K^m \otimes K^n\right]\left[\sum_{0\leq k<N} \frac{1}{[k]_q!}(1-q^{-2})^k q^{k(k+1)/2}\, X_-^k \otimes X_+^k\right]
$$

In the case $N = 3$, the algebra $\mathcal{H}$ has three projective indecomposable modules of dimensions denoted 3_{irr}, 6_{odd} and 6_{eve}. The first one is irreducible whereas the last two are not. The quotient of 6_{odd} (respectively 6_{eve}) by their radical of respective dimensions 5 and 4 give irreducible representations of dimensions 1 and 2. Moreover, the tensor products between irreducible representations and projective indecomposable ones can be reduced as follows:

$$
\begin{array}{llllll}
\underline{2} \times \underline{2} & \equiv & \underline{1} + \underline{3}_{irr} & \underline{6}_{eve} \times \underline{3}_{irr} & \equiv & \underline{6}_{eve} + \underline{6}_{eve} + \underline{3}_{irr} + \underline{3}_{irr} \\
\underline{2} \times \underline{3}_{irr} & \equiv & \underline{6}_{eve} & \underline{6}_{odd} \times \underline{3}_{irr} & \equiv & \underline{6}_{eve} + \underline{6}_{eve} + \underline{3}_{irr} + \underline{3}_{irr} \\
\underline{3}_{irr} \times \underline{3}_{irr} & \equiv & \underline{6}_{odd} + \underline{3}_{irr} & \underline{6}_{eve} \times \underline{6}_{eve} & \equiv & 4(\underline{6}_{eve}) + 4(\underline{3}_{irr}) \\
\underline{6}_{eve} \times \underline{2} & \equiv & \underline{6}_{odd} + \underline{3}_{irr} + \underline{3}_{irr} & \underline{6}_{eve} \times \underline{6}_{odd} & \equiv & 4(\underline{6}_{eve}) + 4(\underline{3}_{irr}) \\
\underline{6}_{odd} \times \underline{2} & \equiv & \underline{6}_{eve} + \underline{3}_{irr} + \underline{3}_{irr} & \underline{6}_{odd} \times \underline{6}_{odd} & \equiv & 2(\underline{6}_{odd}) + 2(\underline{6}_{eve}) + 4(\underline{3}_{irr})
\end{array}
$$

Notice that products of irreducible representations are not always direct sums of irreducibles (this is not a modular category). One can define a concept of truncated (or fusion) tensor product by using the notion of quantum trace and discarding those representations of q-dimension zero. The algebra $\mathcal{H}$ is indeed a Ribbon Hopf algebra and the notion of quantum trace (and of quantum dimension of representations) makes sense. This quantum dimension Tr_q has the property of being multiplicative with respect to tensor products. It can be seen that $Tr_q(X) = Tr(KX)$ so that the q-dimension of the projective indecomposable representations vanishes, whereas it is equal to the q-number $[n]$ for the irreducible representations of (usual) dimensions n. Notice that for each value of q being a primitive N-th root of unity (N odd), there exists a particular projective indecomposable representation of (usual) dimension N which is, at the same time, irreducible; the q dimension of this particular irreducible representation vanishes. For $N = 3$, for instance, one can check, using the Appendix E of [8] that the q-dimension of the projective indecomposable representations (the 3_{irr}, 6_{eve} and 6_{odd}) vanishes, whereas it is equal respectively to 1 and -1 for the irreducible representations of (usual) dimensions 1 and 2.

The previous table of results for tensor products of representations of $\mathcal{H}$ was obtained in [8] without using knowledge of the R-matrix and without using the concept of q-dimension (or truncated tensor product).

5 Reality structures

5.1 Real forms and stars on quantum groups

A $*$-Hopf algebra $\mathcal{F}$ is an associative algebra that satisfies the following properties (for all elements a, b in $\mathcal{F}$):

1. $\mathcal{F}$ is a Hopf algebra (a quantum group), with coproduct Δ, antipode S and counit ϵ.

2. $\mathcal{F}$ is an involutive algebra, i.e., it has an involution $*$ (a 'star' operation). This operation is involutive ($* * a = a$), antilinear ($*(\lambda a) = \overline{\lambda} * a$, where λ is a complex number), and anti-multiplicative ($*(ab) = (*b)(*a)$).

3. The involution is compatible with the coproduct, in the following sense: if $\Delta a = a_1 \otimes a_2$, then $\Delta * a = *a_1 \otimes *a_2$.

4. The involution is also compatible with the counit: $\epsilon(*a) = \overline{\epsilon(a)}$.

5. The following relation with the antipode holds: $S * S * a = a$.

Actually, the last relation is a consequence of the others. It can also be written $S* = *S^{-1}$. It may happen that $S^2 = id$, in which case $S* = *S$, but it is not generally so (usually the square of the antipode is only a conjugacy).

If one wishes, using the $*$ on $\mathcal{F}$, one can define a star operation on the tensor product $\mathcal{F} \otimes \mathcal{F}$, by $*(a \otimes b) = *a \otimes *b$. The third condition reads then $\Delta* = *\Delta$, so one can say Δ is a $*$-homomorphism between $\mathcal{F}$ and $\mathcal{F} \otimes \mathcal{F}$ (each with its respective star). It can also be said that ϵ is a $*$-homomorphism between $\mathcal{F}$ and $\mathbb{C}$ with the star given by complex conjugation.

A star operation as above, making the Hopf-algebra a $*$-Hopf algebra, is also called a *real form* for the given algebra. An element u that is such that $*u = u$ is called a real element.

5.1.1 Twisted stars on quantum groups

It may happen that one finds an involution $*$ on a Hopf algebra for which the third axiom fails in a special way, namely, the case where $\Delta a = a_1 \otimes a_2$ but where $\Delta * a = *a_2 \otimes *a_1$. In this case $S* = *S$. Such an involution is called a *twisted star* operation. Remember that, whenever one has a coproduct Δ on an algebra, it is possible to construct another coproduct Δ^{op} by composing the first one with the tensorial flip. If one defines a star operation on the tensor product (as before) by $*(a \otimes b) \doteq *a \otimes *b$, the property defining a twisted star reads

$$\Delta* = *\Delta^{op} .$$

One should be cautious: some authors introduce a different star operation on the tensor product, namely $*'(a \otimes b) \doteq *b \otimes *a$. In that case, a twisted star operation obeys $\Delta* = *'\Delta$! Twisted star operations are often used in conformal field theory ([14]).

5.1.2 Remark about superstars on differential algebras

On a real manifold, the star operation has a "strange property". Indeed, it is natural to take $*x = x$, $*y = y$ (on the algebra of functions) and extend it to the algebra of differential forms by requiring that the star is that $*dx = dx$, so that it is "real" in the sense of not being complex. However, antimultiplicativity of the star leads immediately to $*(dx\,dy) = (*dy)(*dx) = dy\,dx = -dx\,dy$, so that $dx\,dy$ cannot be a "real element" for this reasonable star! This strange feature does not arise when we stop at the level of the algebra of functions but it shows up as soon as we want to promote a given star to the $\mathbb{Z}_2$-graded algebra of differential forms, something that we shall do later in our context. In order to solve this "problem", which already appears on a usual manifold, it is always possible —but not necessary— to introduce a superstar (nothing to do with the twist described in the previous subsection), i.e., a $\mathbb{Z}_2$-graded star, with the constraint:

$$*(a\,b) = (-1)^{(\#a\#b)} *b *a$$

where $\#a$ is the $\mathbb{Z}_2$-parity of a. Using such superstars allow one to identify "real elements" (in the usual sense of not being complex) with real elements for the $*$ (i.e., such that $*u = u$).

5.2 Real forms on $\mathcal{F}$

As can be easily found in the literature, one has three possibilities for the —not twisted— star operations on $Fun(SL_q(2, \mathbb{C}))$ (up to Hopf automorphisms):

1. The real form $Fun(SU_q(2))$. The matrix elements obey $a^* = d$, $b^* = -qc$, $c^* = -q^{-1}b$ and $d^* = a$. Moreover, q should be real.

2. The real form $Fun(SU_q(1, 1))$. The matrix elements obey $a^* = d$, $b^* = qc$, $c^* = q^{-1}b$ and $d^* = a$. Moreover, q should be real.

3. The real form $Fun(SL_q(2, \mathbb{R}))$. The matrix elements obey $a^* = a$, $b^* = b$, $c^* = b$ and $d^* = d$. Here q can be complex but it should be a phase.

Therefore, taking q a root of unity is incompatible with the SU_q real forms, and the only possibility is to choose the star corresponding to $Fun(SL_q(2, \mathbb{R}))$. This already tells us that there is at most one real form on its quotient $\mathcal{F}$. We only have to check that the star operation preserves the ideal

and coideal defined by $a^3 = d^3 = \mathbb{1}$, $b^3 = c^3 = 0$. This is trivial because $a^* = a$, $b^* = b$, $c^* = c$ and $d^* = d$. Hence, this star operation can be restricted to $\mathcal{F}$.

This real form can be considered as a reduced quantum group associated with the real form $Fun(SL_q(2,\mathbb{R}))$ of $Fun(SL_q(2,\mathbb{C}))$.

Of course, one can also discuss *twisted* star operations: see, in particular the comment at the end of the next subsection.

5.3 Real structures and star operations on $\mathcal{M}$ and $\mathcal{H}$

Now we want to introduce an involution (a star operation) on the comodule algebra $\mathcal{M}$. This involution should be compatible with the coaction of $\mathcal{F}$. That is, we are asking for covariance of the star under the (right,left) $\mathcal{F}$-coaction,

$$(\Delta_{R,L} \, z)^* = \Delta_{R,L}(z^*) \, , \quad \text{for any } z \in \mathcal{M} \, . \tag{21}$$

Here we have used the same notation $*$ for the star on the tensor products, which are defined as $(A \otimes B)^* = A^* \otimes B^*$. Using, for instance, the left coaction in (21), we see immediately that the real form $Fun(SL_q(2,\mathbb{R}))$ corresponds to choosing on $\mathcal{M} = M_N(\mathbb{C})$ the star given by

$$x^* = x \, , \qquad y^* = y \, . \tag{22}$$

We now want to find a compatible $*$ on the algebra $\mathcal{H}$. As $\mathcal{H}$ is dual to $\mathcal{F}$ (or $U_q(sl(2))$ dual to $Fun(SL_q(2,\mathbb{C}))$), we should have dual $*$-structures. This means the relation

$$< h^*, u > = \overline{< h, (Su)^* >} \, , \quad h \in \mathcal{H}, u \in \mathcal{F} \tag{23}$$

holds. In this way we obtain:

$$X_+^* = -q^{-1}X_+ \, , \quad X_-^* = -qX_- \, , \quad K^* = K \, . \tag{24}$$

Moreover, the covariance condition for the star, equation (21), may also be written dually as a condition for $*$ under the action of $\mathcal{H}$. This can be done pairing the $\mathcal{F}$ component of equation (21) with some $h \in \mathcal{H}$. One gets finally the constraint on $*_{\mathcal{H}}$ to be $\mathcal{H}$ covariant,

$$h(z^*) = [(Sh)^*z]^* \, , \quad h \in \mathcal{H}, z \in \mathcal{M} \, . \tag{25}$$

Adding the non quadratic relations $x^N = \mathbb{1}$, $y^N = \mathbb{1}$ in $\mathcal{M}$, and the corresponding ones in the algebra $\mathcal{H}$, does not change anything to the determination of the star structures. This is because the (co)ideals are preserved by the involutions, and thus the quotients can still be done.

Remark that the set of $N \times N$ matrices is endowed with a usual star operation, the hermitian conjugacy †. It is clear that x and y are unitary elements with respect to †: $x^\dagger = x^{-1}$ ($= x^2$ if $N = 3$) and $y^\dagger = y^{-1}$ ($= y^2$ if $N = 3$). But this is *not* the star operation that we are using now, the one that is compatible with the quantum group action, at least when x and y are represented by the 3×3 matrices given in Section 2.

Note finally that one could be tempted to chose the involution defined by $K^\star = K^{-1}$, $X_+^\star = \pm X_-$ and $X_-^\star = \pm X_+$. However, this is a *twisted* star operation. This last operation is the one one would need to interpret the unitary group of $\mathcal{H}$, in the case $N = 3$, as $U(3) \times U(2) \times U(1)$, which could be related to the gauge group of the Standard Model [12].

5.4 A quantum group invariant scalar product on the space of $N \times N$ matrices

5.4.1 Identification between star operation and adjoint

A possible scalar product on the space $\mathcal{M} = M_N(\mathbb{C})$ of $N \times N$ matrices is the usual one, namely, $m_1, m_2 \to Tr(m_1^\dagger m_2)$. For every linear operator ℓ acting on $\mathcal{M}$ we can define the usual adjoint $\ell^\dagger$; however, this adjoint does not coincide with the star operation introduced previously. Our aim in this section is to find another scalar product better suited for our purpose.

We take $z, z' \in \mathcal{M}$, and $h \in \mathcal{H}$. We know that the first ones act on $\mathcal{M}$ like multiplication operators and that h acts on $\mathcal{M}$ by twisted derivations or automorphisms. We also know the action of our star operation $*$ on these linear operators. We shall now obtain our scalar product by imposing that $*$ coincides with the adjoint associated with this scalar product (compatibility condition). That is, we are asking for an inner product on $\mathcal{M}$ such that the actions of $\mathcal{M}$ and $\mathcal{H}$ (each with its respective star) on that vector space may be thought as $*$-representations. Due to the fact that $(z, z') = (\mathbb{1}, z^* z')$, it is enough to compute $(\mathbb{1}, z)$ for all the z belonging to $\mathcal{M}$. The above compatibility condition leads to a single solution [8]: the only non vanishing scalar product between elements of the type $(\mathbb{1}, z)$ is $(\mathbb{1}, x^{N-1} y^{N-1})$. From this quantity we deduce $N^2 - 1$ other non-zero values for the scalar products (z, z') where z and z' are basis elements $x^r y^s$. For instance, $(x, x^{N-2} y^2) = (\mathbb{1}, x^* x^{N-2} y^{N-1}) = (\mathbb{1}, x^{N-1} y^{N-1})$.

Hermiticity with respect to $*$ implies that (xy, xy) should be real, so we set $(xy, xy) = 1$.

5.4.2   Quantum group invariance of the scalar product

We should now justify why the above scalar product was called a quantum group invariant one. Remember we only said the scalar product was such that the stars coincide with the adjoint operators, or such that the actions are given by $*$-representations.

We refer the reader to [6], where it is shown that the $*$-representation condition on the scalar product,

$$(hz, w) = (z, h^* w) \,, \quad h \in \mathcal{H} \,, \tag{26}$$

automatically fulfills one of the two alternative invariance conditions that can be imposed on the scalar product, namely

$$((Sh_1)^* z, h_2 w) = \epsilon(h)(z, w) \,, \quad \text{with } \Delta h = h_1 \otimes h_2 \,. \tag{27}$$

The relations dual to (27) and (26) are those that apply to the coaction of $\mathcal{F}$ instead of the action of $\mathcal{H}$. These are

$$(\Delta_R \, z, \Delta_R \, w) = (z, w) \mathbb{1}_\mathcal{F} \,, \tag{28}$$

where $(\Delta_R \, z, \Delta_R \, w)$ should be understood as $(z_i, w_j) \, T_i^* \, T_j$ if $\Delta_R \, z = z_i \otimes T_i$, and

$$(z, \Delta_R w) = ((1 \otimes S) \Delta_R z, w) \,, \tag{29}$$

respectively. We have used here the right-coaction, but the formulas for the left coaction can be trivially deduced from the above ones.

It is worth noting that these equations for the coaction of $\mathcal{F}$ *imply* the previous ones for the action of $\mathcal{H}$, and are completely equivalent assuming non-degeneracy of the pairing $< \cdot, \cdot >$. Moreover, (28) is a requirement analogous to the condition of classical invariance by a group element action.

Using the unique Hopf compatible star operation $*$ on $\mathcal{H}$, we can calculate the most general metric on the vector spaces of each of the indecomposable representations of $\mathcal{H}$. Obviously, one should restrict the inner product to be a quantum group invariant one. This is done in [8], where we refer for details. Here it suffices to say that one obtains for the projective indecomposable modules nondegenerate but indefinite metrics, and the submodules carry a metric which is, moreover, degenerate.

6   The Manin dual $\mathcal{M}^!$ of $\mathcal{M}$

Our algebra $\mathcal{M}$ is not quadratic since we impose the relations $x^N = y^N = \mathbb{1}$. Nevertheless, we define its Manin dual $\mathcal{M}^!$ as the algebra generated *over the complex numbers* by ξ^1, ξ^2, satisfying the relations

$$(\xi^1)^2 = 0 \,, \quad q \xi^1 \xi^2 + \xi^2 \xi^1 = 0 \,, \quad (\xi^2)^2 = 0 \,,$$

as in the unreduced case. We shall write $dx \doteq \xi^1$ and $dy \doteq \xi^2$, so $\mathcal{M}^!$ is defined by these two generators and the relations

$$dx^2 = 0 \,, \quad dy^2 = 0 \,, \quad q dx \, dy + dy \, dx = 0 \,. \tag{30}$$

Once the coaction of $\mathcal{F}$ on $\mathcal{M}$ has been defined as in Section 3, it is easy to check that its coaction on $\mathcal{M}^!$ is given by the same formulae. Namely, writing

$$\begin{pmatrix} dx' \\ dy' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} dx \\ dy \end{pmatrix}$$

and

$$(\tilde{d}x \quad \tilde{d}y) = (dx \quad dy) \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

ensures that $q \, dx' \, dy' + dy' \, dx' = 0$ and $q \, \tilde{d}x \, \tilde{d}y + \tilde{d}y \, \tilde{d}x = 0$, once the relation $q \, dx \, dy + dy \, dx = 0$ is satisfied. The left and right coactions can be read from those formulae, for instance $\Delta_R(dx) = dx \otimes a + dy \otimes c$.

Since the formulae for the coactions on the generators and on their differentials are the same, the formulae for the actions of $\mathcal{H}$ on $\mathcal{M}^!$ must also coincide. For instance, using $X_-^L[x] = y$ we find immediately $X_-^L[dx] = dy$. This corresponds to an irreducible two dimensional representation of $\mathcal{H}$. We shall return to this problem in the next section, since we are going to analyse the decomposition in representations of a differential algebra $\Omega_{WZ}(\mathcal{M})$ built using $\mathcal{M}^!$.

7 Covariant differential calculus on $\mathcal{M}$

Given an algebra $\mathcal{A}$, there is a universal construction that allows one to build the so-called *algebra of universal differential forms* $\Omega(\mathcal{A}) = \sum_{p=0}^{\infty} \Omega^p(\mathcal{A})$ over $\mathcal{A}$. This differential algebra is universal, in the sense that any other differential algebra with $\Omega^0(\mathcal{A}) = \mathcal{A}$ will be a quotient of the universal one. For practical purposes, it is often not very convenient to work with the algebra of universal forms. First of all, it is very "big". Next, it does not remember anything of the coaction of $\mathcal{F}$ on the algebra $\mathcal{M}$ (the 0-forms).

Starting from a given algebra, there are several constructions that allow one to build "smaller" differential calculi. As already mentioned, they will all be quotients of the algebra of universal forms by some (differential) ideal. One possibility for such a construction was described by [15], another one by [16], and yet another one by [17]. In the present case, however, we use something else, namely the differential calculus Ω_{WZ} introduced by Wess and Zumino [18] in the case of the quantum 2-plane. Its main interest is that it is covariant with respect to the action (or coaction) of a quantum group. Its construction was recalled in [6] where it was also shown that one can further take another quotient by a differential ideal associated with the constraints $x^N = y^N = \mathbb{1}$ (so that, indeed $d(x^N) = d(y^N) = 0$ automatically).

7.1 The reduced Wess-Zumino complex

The algebra Ω_{WZ} is a differential algebra first introduced by [18] for the "full" quantum plane. First of all $\Omega_{WZ} = \Omega_{WZ}^0 \oplus \Omega_{WZ}^1 \oplus \Omega_{WZ}^2$ is a graded vector space.

- Forms of grade 0 are just functions on the quantum plane, i.e., elements of $\mathcal{M}$.
- Forms of grade 1 are of the type $a_{rs} x^r y^s dx + b_{rs} x^r y^s dy$, where dx and dy are the generators of the Manin dual $\mathcal{M}^!$.
- Forms of grade 2 are of the type $c_{rs} x^r y^s dx \, dy$.

Next, Ω_{WZ} is an algebra. The relations between x, y, dx and dy are determined by covariance under the quantum group action:

$$
\begin{aligned}
& xy = qyx \\
& x \, dx = q^2 dx \, x && x \, dy = q \, dy \, x + (q^2 - 1) dx \, y \\
& y \, dx = q \, dx \, y && y \, dy = q^2 dy \, y && (31) \\
& dx^2 = 0 && dy^2 = 0 \\
& dx \, dy + q^2 dy \, dx = 0
\end{aligned}
$$

Moreover, we want Ω_{WZ} to be a differential algebra, so we introduce an operator d and set $d(x) = dx$, $d(y) = dy$; the Leibniz rule (for d) should also hold. Finally, we impose $d\mathbb{1} = 0$ and $d^2 = 0$.

In the case $q^N = 1$, we add to Ω_{WZ} the extra defining relation (coming from the definition of the reduced quantum plane): $x^N = y^N = \mathbb{1}$. The fact that Ω_{WZ} is still well defined as a differential algebra is not totally obvious and requires some checking (see [8]). Note that $\dim(\Omega^0_{WZ}) = N^2$, $\dim(\Omega^1_{WZ}) = 2N^2$ and $\dim(\Omega^2_{WZ}) = N^2$.

7.2 The action of $\mathcal{H}$ on $\Omega_{WZ}(\mathcal{M})$

Since $\mathcal{H}$ acts on $\mathcal{M}$ (and we know how this module decomposes under the action of $\mathcal{H}$), it is clear that we can also decompose Ω_{WZ} in representations of $\mathcal{H}$. This was done explicitly (for the case $N = 3$) in [6], and the action of $\mathcal{H}$ on Ω^1_{WZ} (for an arbitrary N) was described in [7]. The cohomology of d is actually non trivial and was also studied in [8].

7.3 The space of differential operators on $\mathcal{M}$

We now summarize the structure of the space of differential operators on the reduced quantum plane $\mathcal{M}$, i.e., the algebra of $N \times N$ matrices.

The space $\mathcal{D}$ of differential operators on $\mathcal{M}$

We already know what the operator $d : \Omega^0_{WZ} = \mathcal{M} \to \Omega^1_{WZ}$ is. *A priori* $df = dx\, \partial_x(f) + dy\, \partial_y(f)$, where $\partial_x f, \partial_y f \in \mathcal{M}$, and this defines ∂_x and ∂_y as (linear) operators on $\mathcal{M}$.

Generally speaking, operators of the type $f(x,y)\partial_x$ or $f(x,y)\partial_y$ are called differential operators of order 1. Composition of such operators gives rise to differential operators of order higher than 1. Multiplication by elements of $\mathcal{M}$ is considered as a differential operator of degree 0. The space of all these operators is a vector space $\mathcal{D} = \oplus_{i=0}^{i=4}\mathcal{D}_i$.

The twisting automorphisms σ^3

Since we know how to commute x, y with dx, dy, we can write, for any element $f \in \mathcal{M}$

$$
\begin{aligned}
f\, dx &= dx\, \sigma^x_x(f) + dy\, \sigma^x_y(f) \\
f\, dy &= dx\, \sigma^y_x(f) + dy\, \sigma^y_y(f)
\end{aligned}
$$

where each σ^i_j is an element of $End(\mathcal{M})$ to be determined (just take $f = x$ and $f = y$ in the above to get the results that one is looking for).

Let f and g be elements of $\mathcal{M}$. From the associativity property $(fg)dz = f(gdz)$ we find

$$
\sigma^j_i(fg) = \sigma^k_i(f)\, \sigma^j_k(g)
$$

with a summation over the repeated indices. The map $\sigma : f \in \mathcal{M} \to \sigma(f) \in M_2(\mathcal{M})$ is an algebra homomorphism from the algebra $\mathcal{M}$ to the algebra $M_2(\mathcal{M})$ of 2×2 matrices, with elements in $\mathcal{M}$.

The usual Leibniz rule for d, namely $d(fg) = d(f)g + fd(g)$, implies that

$$
\partial_i(fg) = \partial_i(f)\, g + \sigma^j_i(f)\, \partial_j(g) \ .
$$

This shows that ∂_x and ∂_y are derivations twisted by an automorphism.

Relations in $\mathcal{D}$

For calculational purposes, it is useful to know the commutation relations between x, y and ∂_x, ∂_y, those between ∂_x, ∂_y and σ^j_i and the relations between the σ^j_i. Here are some results (see also [18, 10]).

$$
\begin{aligned}
\partial_x x &= 1 + q^2 x\, \partial_x + (q^2 - 1)y\, \partial_y \\
\partial_x y &= qy\, \partial_x
\end{aligned}
$$

$$
\begin{aligned}
\partial_y x &= qx\, \partial_y \\
\partial_y y &= 1 + q^2 y\, \partial_y
\end{aligned}
$$

---

[3] Some properties of these automorphisms are discussed in [19]

Moreover,
$$\partial_y\,\partial_x = q\partial_x\,\partial_y$$

Also, the relations $x^N = y^N = \mathbb{1}$ lead to other constraints on the powers of the derivations. For example, for $N = 3$ these imply the constraint:
$$\partial_x\partial_x\partial_x = \partial_y\partial_y\partial_y = 0$$

Finally, the commutation relations between the σ's can be calculated from the values of the $\sigma_j^i(x)$.

Differential operators on $\mathcal{M}$ associated with the $\mathcal{H}$ action

The twisted derivations ∂_x, ∂_y considered previously constitute a q-analogue of the notion of vector fields. Their powers build up arbitrary differential operators. Elements of $\mathcal{H}$ act also like powers of generalized vector fields (consider, for instance, the left action generated by $X_\pm^L, K^L$). Of course, they are differential operators of a special kind. One can say that elements of $\mathcal{H}$ act on $\mathcal{M}$ as *invariant* differential operators since they are associated with the action of a (quantum) group on a (quantum) space.

A priori, the generators $X_\pm^L, K^L$ can be written in terms of $x, y, \partial_x, \partial_y$. The coefficients of such combination can be determined by imposing that equations (12), (15) are satisfied. A rather cumbersome calculation leads to a unique solution (cf. [8]) that can be written simply in terms of the scaling operators [20] $\mu_x \equiv \mathbb{1} + (q^2 - 1)(x\partial_x + y\partial_y)$, $\mu_y \equiv \mathbb{1} + (q^2 - 1)(y\partial_y)$:

$$
\begin{aligned}
K_-^L &= \mu_x\mu_y \\
K^L &= \mu_x\mu_x\mu_y\mu_y \\
X_+^L &= q^{-1}x\partial_y\mu_y\mu_y \\
X_-^L &= qy\partial_x\mu_x
\end{aligned}
$$

Notice that elements of $\mathcal{M}$ acting by multiplication on itself can be considered as differential operators of order zero. It makes therefore perfect sense to study the commutation relations between the generators x, y of the quantum plane and $X_\pm, K$. This is also done in [8].

7.4 Star operations on the differential calculus $\Omega_{WZ}(\mathcal{M})$

Given a $*$ operation on the algebra $\mathcal{M}$, we want to extend it to the differential algebra $\Omega_{WZ}(\mathcal{M})$. This can be done in two ways, either by using a usual star operation, or by using a superstar operation (see subsection superstar). Here we use the "usual" star operation formalism, so that the star has to be involutive, complex sesquilinear, and anti-multiplicative for the algebra structure in $\Omega_{WZ}(\mathcal{M})$. We impose moreover that it should be compatible with the coaction of $\mathcal{F}$. The quantum group covariance condition is, again, just the commutativity of the $*$, $\Delta_{R,L}$ diagram, or, algebraically,

$$(\Delta_{R,L}\omega)^* = \Delta_{R,L}(\omega^*) \ . \tag{32}$$

However, there is no reason *a priori* to impose that $*$ should commute with d. In any case, it is enough to determine the action of $*$ on the generators dx and dy, since we already determined the $*$ operation on $\mathcal{M}$ ($*x = x$, $*y = y$).

Taking $\Delta_{R,L}dx = a \otimes dx + b \otimes dy$, we get $(\Delta_{R,L}dx)^* = a \otimes dx^* + b \otimes dy^*$, to be compared with $\Delta_{R,L}(dx^*)$. Expanding dx^* as a generic element of $\Omega_{WZ}^1(\mathcal{M})$ (we want a grade-preserving $*$), it can be seen that the solution $dx^* = dx$ is the only possible one, up to complex phases. Doing the same with dy we get:

$$dx^* = dx \ , \qquad dy^* = dy \ . \tag{33}$$

The star being now defined on $\Omega_{WZ}^0 = \mathcal{M}$ and on the d of the generators of $\mathcal{M}$, it is extended to the whole of the differential algebra Ω_{WZ} by imposing the anti-multiplicative property $*(\omega_1\omega_2) = (*\omega_2)(*\omega_1)$. With this result, it can be checked that

$$d * \omega = (-1)^p * d\omega \quad \text{when} \quad \omega \in \Omega_{WZ}^p \ . \tag{34}$$

13

The above involution is not the only one that one can define on the Wess-Zumino complex. However, any other involution would not be compatible with the coaction of $\mathcal{F}$. Loosing the compatibility with the quantum group is clearly unacceptable, since the main interest of the Wess-Zumino differential complex rests on the fact that it is compatible with the coaction.

8 Non commutative generalized connections on $\mathcal{M}$ and their curvature

Let Ω be a differential calculus over a unital associative algebra $\mathcal{M}$, i.e., a graded differential algebra with $\Omega^0 = \mathcal{M}$. Let $\mathcal{M}$ be a right module over $\mathcal{M}$. A covariant differential ∇ on $\mathcal{M}$ is a map $\mathcal{M} \otimes_{\mathcal{M}} \Omega^p \mapsto \mathcal{M} \otimes_{\mathcal{M}} \Omega^{p+1}$, such that

$$\nabla(\psi\lambda) = (\nabla\psi)\lambda + (-1)^s \psi\, d\lambda$$

whenever $\psi \in \mathcal{M} \otimes_{\mathcal{M}} \Omega^s$ and $\lambda \in \Omega^t$. ∇ is clearly not linear with respect to the algebra $\mathcal{M}$ but it is easy to check that the curvature ∇^2 is a linear operator with respect to $\mathcal{M}$.

In the particular case where the module $\mathcal{M}$ is taken as the algebra $\mathcal{M}$ itself, any one-form ω (any element of Ω^1) defines a covariant differential. One sets simply $\nabla\mathbb{1} = \omega$, where $\mathbb{1}$ is the unit of the algebra $\mathcal{M}$ and we call curvature the quantity $\rho = \nabla^2 \mathbb{1}$,

$$\rho \doteq \nabla\omega = \nabla\mathbb{1}\omega = (\nabla\mathbb{1})\omega + \mathbb{1}d\omega = d\omega + \omega^2 \ .$$

8.1 Connections on $\mathcal{M}$ and their curvature

We now return to the specific case where $\mathcal{M}$ is the algebra of functions over the quantum plane at a N-th root of unity.

The most general connection is defined by an element ϕ of $\Omega^1_{WZ}(\mathcal{M})$. Since we have a quantum group action of $\mathcal{H}$ on Ω_{WZ}, it is convenient to decompose ϕ into representations of this algebra as obtained in Section 7.2. The exact expression of the curvature $\rho = d\phi + \phi\phi$ is not very illuminating but it can be simplified in several particular cases (see [8]).

As we know, the only Hopf star operation compatible with the quantum group action of $\mathcal{H}$ on the differential algebra Ω_{WZ}, when $q^N = 1$, is the one described in Section 7.4 ($dx^* = dx$, $dy^* = dy$). Imposing the hermiticity property $\phi = \phi^*$ on the connection leads to constraints on the coefficients. Again we refer to [8] for a discussion of the results.

9 Incorporation of Space-Time

9.1 Algebras of differential forms over $C^\infty(M) \otimes \mathcal{M}$

Let Λ be the algebra of usual differential forms over a space-time manifold M (the De Rham complex) and $\Omega_{WZ} \doteq \Omega_{WZ}(\mathcal{M})$ the differential algebra over the reduced quantum plane introduced in Section 7. Remember that $\Omega^0_{WZ} = \mathcal{M}$, $\Omega^1_{WZ} = \mathcal{M}\, dx + \mathcal{M}\, dy$, and $\Omega^2_{WZ} = \mathcal{M}\, dx\, dy$. We call Ξ the graded tensor product of these two differential algebras:

$$\Xi \doteq \Lambda \otimes \Omega_{WZ}$$

- A generic element of $\Xi^0 = \Lambda^0 \otimes \Omega^0_{WZ}$ is a 3×3 matrix with elements in $C^\infty(M)$. It can be thought as a scalar field valued in $M_3(\mathbb{C})$.

- A generic element of $\Xi^1 = \Lambda^0 \otimes \Omega^1_{WZ} \oplus \Lambda^1 \otimes \Omega^0_{WZ}$ is given by a triplet $\omega = (A_\mu, \phi_x, \phi_y)$, where A_μ determines a one-form (a vector field) on the manifold M with values in $M_3(\mathbb{C})$ (that we can consider as the Lie algebra of the Lie group $GL(3, \mathbb{C})$), and where ϕ_x, ϕ_y are $M_3(\mathbb{C})$-valued scalar fields. Indeed $\phi_x(x^\mu)\, dx + \phi_y(x^\mu)\, dy \in \Lambda^0 \otimes \Omega^1_{WZ}$.

- A generic element of $\Xi^2 = \Lambda^0 \otimes \Omega^2_{WZ} \oplus \Lambda^1 \otimes \Omega^1_{WZ} \oplus \Lambda^2 \otimes \Omega^0_{WZ}$
  consists of

    - a matrix-valued 2-form $F_{\mu\nu}dx^\mu dx^\nu$ on the manifold M, i.e., an element of $\Lambda^2 \otimes \Omega^0_{WZ}$

– a matrix-valued scalar field on M, i.e., an element of $\Lambda^0 \otimes \Omega^2_{WZ}$
– two matrix-valued vector fields on M, i.e., an element of $\Lambda^1 \otimes \Omega^1_{WZ}$

The algebra Ξ is endowed with a differential (of square zero, of course, and obeying the Leibniz rule) defined by $d \doteq d \otimes id \pm id \otimes d$. Here $\pm$ is the (differential) parity of the first factor of the tensor product upon which d is applied, and the two d's appearing on the right hand side are the usual De Rham differential on antisymmetric tensor fields and the differential of the reduced Wess-Zumino complex, respectively.

If G is a Lie group acting on the manifold M, it acts also (by pull-back) on the functions on M and, more generally, on the differential algebra Λ. For instance, we may assume that M is Minkowski space and G is the Lorentz group. The Lie algebra of G and its enveloping algebra $\mathcal{U}$ also act on Λ, by differential operators. Intuitively, elements of Ξ have an "external" part (i.e., functions on M) on which $\mathcal{U}$ act, and an "internal" part (i.e., elements belonging to $\mathcal{M}$) on which $\mathcal{H}$ acts. We saw that $\mathcal{H}$ is a Hopf algebra (neither commutative nor cocommutative) whereas $\mathcal{U}$, as it is well known, is a non commutative but cocommutative Hopf algebra. To conclude, we have an action of the Hopf algebra $\mathcal{U} \otimes \mathcal{H}$ on the differential algebra Ξ.

9.2   Generalized gauge fields

Since we have a differential algebra Ξ associated with the associative algebra $C^\infty(M) \otimes \mathcal{M}$, we can define, as usual, "abelian"-like connections by choosing a module which is equal to the associative algebra itself. A Yang-Mills potential ω is an arbitrary element of Ξ^1 and the corresponding curvature, $d\omega + \omega^2$, is an element of Ξ^2. Using the results of the previous subsection, we see that $\omega = (A_\mu, \phi_x, \phi_y)$ consists of a usual Yang-Mills field A_μ valued in $M_3(\mathbb{C})$ and a pair ϕ_x, ϕ_y of scalar fields also valued in the space of 3×3 matrices. We have $\omega = A + \phi$, where $A = A_\mu dx^\mu$ and $\phi = \phi_x dx + \phi_y dy \in \Lambda^0 \otimes \Omega^1_{WZ} \subset \Xi^1$. We can also decompose $A = A^\alpha \lambda_\alpha$, with λ_α denoting the usual Gell-Mann matrices (together with the unit matrix) and A^α a set of complex valued one-forms on the manifold M. Let us call δ the differential on Ξ, $\underline{d}$ the differential on Λ and d the differential on Ω_{WZ} (as before). The curvature is then $\delta\omega + \omega^2$. Explicitly,

$$\delta A = (\underline{d}A^\alpha)\lambda_\alpha - A^\alpha d\lambda_\alpha$$

and

$$\delta\phi = (\underline{d}\phi_x)dx + (\underline{d}\phi_y)dy + (d\phi_x)dx + (d\phi_y)dy \ .$$

It is therefore clear that the corresponding curvature will have several pieces:

• The Yang-Mills strength F of A

$$F \doteq (\underline{d}A^\alpha)\lambda_\alpha + A^2 \quad \in \Lambda^2 \otimes \Omega^0_{WZ}$$

• A kinetic term $\mathcal{D}\phi$ for the scalars, consisting of three parts: a purely derivative term, a covariant coupling to the gauge field and a mass term for the Yang-Mills field (linear in the A_μ's).

$$\mathcal{D}\phi \doteq (\underline{d}\phi_x)dx + (\underline{d}\phi_y)dy + A\phi + A^\alpha d\lambda_\alpha \quad \in \Lambda^1 \otimes \Omega^1_{WZ}$$

• Finally, a self interaction term for the scalars

$$(d\phi_x)dx + (d\phi_y)dy + \phi^2 \quad \in \Lambda^0 \otimes \Omega^2_{WZ}$$

Hence we recover the usual ingredients of a Yang-Mills-Higgs model (the mass term for the gauge field, linear in A, is usually obtained from the "$A\phi$ interaction" after shifting ϕ by a constant).

By choosing an appropriate scalar product on the space Ξ^2, one obtains a quantity that is quadratic in the curvatures (quartic in the fields) and that could be a candidate for the Lagrangian of a theory of Yang-Mills-Higgs type. However, if we do not make specific choices for the connection (for instance by imposing reality constraints or by selecting one or another representation of $\mathcal{H}$), the results are a bit too general and, in any case, difficult to interpret physically. Actually, many choices are possible and we do not know, at the present level of our analysis, which kind of constraint could give rise to interesting physics.

10 Concluding remarks

Physical models of the gauge type will involve the consideration of one-forms. If we restrict ourselves to the "internal space" part of these one-forms, we have to consider objects of the form $\Phi = \sum \varphi_i \omega_i$. Here $\{\omega_i\}$ is a basis of some non-trivial indecomposable representation of $\mathcal{H}$ (or any other non-cocommutative quantum group) on the space of 1-forms, and φ_i are functions over some space-time manifold. What about the transformation properties of the fields φ_i? This is a question of central importance, since, ultimately, we will integrate out the internal space (whatever this means), and the only relic of the quantum group action on the theory will be the transformations of the fields φ_i's. There are several possibilities: one of them, as suggested from the results of Section 9 is to consider $\mathcal{H}$ as a discrete analogue of the Lorentz group (actually, of the enveloping algebra $\mathcal{U}$ of its Lie algebra). In such a case, "geometrical quantities", like Φ should be $\mathcal{H}$-invariant (and $\mathcal{U}$-invariant). This requirement obviously forces the φ_i to transform. Another possibility would be to assume that Φ itself transforms according to some representation of this quantum group (in the same spirit one can study, classically, the invariance of particularly chosen connections under the action of a group acting also on the base manifold). In any case, the φ_i are going to span some non-trivial representation space of $\mathcal{H}$.

Usually, the components ϕ_i are real (or complex) numbers and are, therefore, commuting quantities. However, this observation leads to the following problem: If the components of the fields commute, then we get $h.(\varphi_i \varphi_j) = h.(\varphi_j \varphi_i)$, for any $h \in \mathcal{H}$. This would imply (here $\Delta h = h_1 \otimes h_2$)

$$
\begin{aligned}
(h_1.\varphi_i)(h_2.\varphi_j) &= (h_1.\varphi_j)(h_2.\varphi_i) \\
&= (h_2.\varphi_i)(h_1.\varphi_j) \ .
\end{aligned}
$$

This equality cannot be true in general for a non-cocommutative coproduct. Hence we should generally have a nonabelian product for the fields. In our specific case, there is only one abelian $\mathcal{H}$-module algebra, the 3_{odd} one. Only fields transforming according to this representation could have an abelian product. However, covariance strongly restricts the allowable scalar products on each of the representation spaces (for instance, in the case of $\mathcal{H}$ we get both indefinite and degenerate metrics). This fact is particularly important as one should have a positive definite metric on the physical degrees of freedom. To this end one should disregard the non-physical (gauge) ones, and look for representations such that only positive definite states survive. Thus we see that the selection of the representation space upon which to build the physical model is not simple.

The fact of having noncommuting fields has a certain resemblance with the case of supersymmetry. As the superspace algebra is noncommutative, the scalar superfield must have noncommuting component fields in order to match its transformation properties. As a consequence, instead of having —on each space-time point— just the Grassmann algebra over the complex numbers, we see the appearance of an enlarged algebra generated by both the variables and the fields. It is reasonable to expect that the addition of a non-trivial quantum group as a symmetry of space forces a more constrained algebra.

We should point out that the above reasoning is very general, and is independent of the details of the fields. That is, it relies only in the existence of a non-cocommutative Hopf algebra acting in a nontrivial way on the fields.

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